Structural Operational Semantics with transitivity rules and execution time

Patricia Peratto
psperatto@adinet.com.uy
Tecnólogo en Informática
FING-CETP.UdelaR-ANEP

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Abstract

We define an structural operational semantics of the core of an imperative language. It has a measure in the transitions corresponding to the number of steps of evaluation that takes place in the transition (from the point of view of usual complexity theory) and transitivity rules that allow to prove in the theory what is usually proved in the meta-theory.

1 Introduction

Our semantics has transitions of the form $\Rightarrow^k$ where k is the number of steps of evaluation that takes place in the transition.

Rules to measure the time of evaluation of arithmetic expressions and statements for the natural semantics have been given in [3]. To our knowledge, haven't been presented for the structural operational semantics.

This semantics applies to the evaluation of expressions and statements and can allow to measure the complexity of a program at the same time that we derive its construction applying the rules. The semantics is compositional and the measure of time also.

We show our semantics is deterministic and equivalent to the structural operational semantics without transitivity rules. We also show it is equivalent to the natural semantics with exact execution times presented in [3].

The outline of the paper is as follows: in section 2 we present the abstract syntax of the language and we introduce categories for different kind of expressions and for statements; follows in section 3 the semantics of the expressions and the set of rules for the semantics of statements. In section 4 we prove that the semantics is deterministic. In section 5 we present the equivalence with structural operational semantics. In section 6 we present the equivalence with natural semantics with execution times. In section 7 we present conclusions and further work.

2 Abstract syntax of the language

We use a syntactic notation based on BNF and we will use parenthesis (not indicated in our BNF) to solve ambiguities and uniquely determine the corresponding parse tree.

We have the following Syntactic Categories and meta-variables ranging over them:

n will range over numerals, **Num**
 ax will range over arithmetic variables, **AVar**
 a will range over arithmetic expressions, **AExp**
 b will range over boolean expressions, **BExp**
 S will range over statements, **Stm**

The meta-variables can be primed or subscripted for example n, n', n_1, n_2 all stand for numerals.

2.1 Abstract syntax for arithmetic variables

$x ::= \text{letter } lx$
 $lx ::= \epsilon \mid \text{letter } lx \mid \text{digit } lx$

That is, an string of letters and digits beginning with a letter. We assume letter and digit as understood.

2.2 Abstract syntax for Arithmetic Expressions

$a ::= n \mid ax \mid a_1 + a_2 \mid a_1 * a_2 \mid a_1 - a_2$

2.3 Abstract syntax for Boolean Expressions

$b ::= \text{true} \mid \text{false} \mid a_1 = a_2 \mid a_1 < a_2 \mid \neg b \mid b_1 \vee b_2 \mid b_1 \wedge b_2$

2.4 Abstract syntax for Statements

$S ::= ax := a \mid \text{skip} \mid S_1; S_2 \mid \text{if } b \text{ then } S_1 \text{ else } S_2 \mid \text{while } b \text{ do } S$

3 Measured semantics of expressions

We define the following semantic functions:

$A : \mathbf{Aexp} \rightarrow (\mathbf{State} \rightarrow \mathbf{Z} \times \mathbf{N})$ for arithmetic expressions
 $B : \mathbf{Bexp} \rightarrow (\mathbf{T} \times \mathbf{N})$ for boolean expressions

where

$\mathbf{N}$ is the semantic domain of natural numbers,
 $\mathbf{Z}$ is the semantic domain of integers,
 $\mathbf{T}$ is the semantic domain of truth values,

and

$\mathbf{State} = \mathbf{AVar} \rightarrow \mathbf{Z}$

We have the following metavariables ranging over semantic categories: $k \in \mathbf{N}$, $z \in \mathbf{Z}$, $t \in \mathbf{T}$ and $s \in \mathbf{State}$ primed or subscripted.

3.1 Semantic of Arithmetic Expressions

We define a transition system with transitions of the form

$$\langle a, s \rangle \rightarrow^k z$$

We define the following semantic functions

$$A(a)s = \langle z, k \rangle \text{ if exists } k \text{ such that } \langle a, s \rangle \rightarrow^k z$$

$$Value(a)s = first(A(a)s)$$

$$Time(a)s = second(A(a)s)$$

Transition system:

$$\langle n, s \rangle \rightarrow^1 n$$

$$\langle ax, s \rangle \rightarrow^1 s(ax)$$

$$\frac{\langle a_1, s \rangle \rightarrow^{k_1} z_1 \quad \langle a_2, s \rangle \rightarrow^{k_2} z_2}{\langle a_1 + a_2, s \rangle \rightarrow^{k_1+k_2+1} z_1 + z_2}$$

$$\frac{\langle a_1, s \rangle \rightarrow^{k_1} z_1 \quad \langle a_2, s \rangle \rightarrow^{k_2} z_2}{\langle a_1 * a_2, s \rangle \rightarrow^{k_1+k_2+1} z_1 * z_2}$$

$$\frac{\langle a_1, s \rangle \rightarrow^{k_1} z_1 \quad \langle a_2, s \rangle \rightarrow^{k_2} z_2}{\langle a_1 - a_2, s \rangle \rightarrow^{k_1+k_2+1} z_1 - z_2}$$

Theorem 3.1. *A, Value and Time are functions.*

Given a and s, there exists only one k and z such that $\langle a, s \rangle \rightarrow^k z$.

Proof. By structural induction on a. □

3.2 Semantic of Boolean Expressions

We define a transition system with transitions of the form

$$\langle b, s \rangle \rightarrow^k t$$

We define the following semantic functions

$$B(b)s = \langle t, k \rangle \text{ if exists } k \text{ such that } \langle b, s \rangle \rightarrow^k t$$

$$Value(b)s = first(B(b)s)$$

$$Time(b)s = second(B(b)s)$$

Transition system:

$$\langle true, s \rangle \rightarrow^1 true$$

$$\langle false, s \rangle \rightarrow^1 false$$

$$\frac{A(a_1)s = \langle z_1, k_1 \rangle \quad A(a_2)s = \langle z_2, k_2 \rangle}{\langle a_1 = a_2, s \rangle \rightarrow^{k_1+k_2+1} z_1 = z_2}$$

$$\frac{A(a_1)s = \langle z_1, k_1 \rangle \quad A(a_2)s = \langle z_2, k_2 \rangle}{\langle a_1 < a_2, s \rangle \rightarrow^{k_1+k_2+1} z_1 < z_2}$$

$$\frac{\langle b, s \rangle \rightarrow^k t}{\langle \neg b, s \rangle \rightarrow^{k+1} \neg t}$$

$$\frac{\langle b_1, s \rangle \rightarrow^{k_1} t_1 \quad \langle b_2, s \rangle \rightarrow^{k_2} t_2}{\langle b_1 \vee b_2, s \rangle \rightarrow^{k_1+k_2+1} t_1 \vee t_2}$$

$$\frac{\langle b_1, s \rangle \rightarrow^{k_1} t_1 \quad \langle b_2, s \rangle \rightarrow^{k_2} t_2}{\langle b_1 \wedge b_2, s \rangle \rightarrow^{k_1+k_2+1} t_1 \wedge t_2}$$

Theorem 3.2. *B, Value and Time are functions.*

Given b and s, there exists only one k and t such that $\langle b, s \rangle \rightarrow^k t$.

Proof. By structural induction on b. □

3.3 Semantics of statements

The set of rules corresponding to statements is based in the rules for the Structural Operational Semantics of While presented in [3]. Are new the rules for transitivity and the rules for time. As is proved after, the rules for time are equivalent to the ones presented for natural semantics in the same book.

$$\langle ax := a, s \rangle \Rightarrow^{Time(a)s+1} s[ax := Value(a)s] \quad ass1$$

$$\langle skip, s \rangle \Rightarrow^1 s \quad skip$$

$$\frac{\langle S_1, s \rangle \Rightarrow^k \langle S'_1, s' \rangle}{\langle S_1; S_2, s \rangle \Rightarrow^k \langle S'_1; S_2, s' \rangle} \quad comp1$$

$$\frac{\langle S_1, s \rangle \Rightarrow^k s'}{\langle S_1; S_2, s \rangle \Rightarrow^k \langle S_2, s' \rangle} \quad comp2$$

$$\frac{Value(b)s = true}{\langle if b then S_1 else S_2, s \rangle \Rightarrow^{Time(b)s+1} \langle S_1, s \rangle} \quad if1$$

$$\frac{Value(b)s = false}{\langle if b then S_1 else S_2, s \rangle \Rightarrow^{Time(b)s+1} \langle S_2, s \rangle} \quad if2$$

$$\langle while b do S, s \rangle \Rightarrow^1 \langle if b then (S; while b do S) else skip, s \rangle \quad while1$$

$$\frac{\langle S, s \rangle \Rightarrow^{k_1} \langle S', s' \rangle \quad \langle S', s' \rangle \Rightarrow^{k_2} \langle S'', s'' \rangle}{\langle S, s \rangle \Rightarrow^{k_1+k_2} \langle S'', s'' \rangle} \quad trans1$$

$$\frac{\langle S, s \rangle \Rightarrow^{k_1} \langle S', s' \rangle \quad \langle S', s' \rangle \Rightarrow^{k_2} s''}{\langle S, s \rangle \Rightarrow^{k_1+k_2} s''} \text{ trans2}$$

We define the semantic function

$S_{tos}(S)s = s'$ in k steps if and only if $\langle S, s \rangle \Rightarrow^k s'$

4 Determinism

Theorem 4.1 (The measured semantics presented in subsection 3.3 is deterministic.). *If $\langle S, s \rangle \Rightarrow^k \langle S', s' \rangle$ and $\langle S, s \rangle \Rightarrow^k \langle S'', s'' \rangle$ then $S' = S''$ and $s' = s''$ and if $\langle S, s \rangle \Rightarrow^k s'$ and $\langle S, s \rangle \Rightarrow^k s''$ then $s' = s''$.*

Proof. By induction on the shape of the derivation tree. If the statement is assignment, skip, if or while there is only one possible transition.

Assume the statement is $S_1; S_2$ (composition).

- suppose $\langle S_1; S_2, s \rangle \Rightarrow^k \langle S_3, s' \rangle$ and $\langle S_1; S_2, s \rangle \Rightarrow^k \langle S_4, s'' \rangle$ then the form of S_3 and S_4 is $S_3 = S'_1; S_2$ and $S_4 = S''_1; S_2$ necessarily follows that the hypothesis of the rule are $\langle S_1, s \rangle \Rightarrow^k \langle S'_1, s' \rangle$ or $\langle S_1, s \rangle \Rightarrow^k \langle S''_1, s'' \rangle$. Then by inductive hypothesis $S'_1 = S''_1$ and $s' = s''$.
- Suppose $\langle S_1; S_2, s \rangle \Rightarrow^k \langle S_2, s' \rangle$ and $\langle S_1; S_2, s \rangle \Rightarrow^k \langle S_2, s'' \rangle$ necessarily follows that the hypothesis of the rule are $\langle S_1, s \rangle \Rightarrow^k s'$ or $\langle S_1, s \rangle \Rightarrow^k s''$. Then by inductive hypothesis $s' = s''$.

Assume the rule applied is trans1.

- suppose $\langle S, s \rangle \Rightarrow^{k_1+k_2} \langle S_1, s_1 \rangle$ and $\langle S, s \rangle \Rightarrow^{k_1+k_2} \langle S_2, s_2 \rangle$. The hypothesis of the rule must be of the form $\langle S, s \rangle \Rightarrow^{k_1} \langle S', s' \rangle$ and $\langle S', s' \rangle \Rightarrow^{k_2} \langle S_1, s_1 \rangle$ or $\langle S', s' \rangle \Rightarrow^{k_2} \langle S_2, s_2 \rangle$. Follows by the inductive hypothesis $S_1 = S_2$ and $s_1 = s_2$.
- suppose $\langle S, s \rangle \Rightarrow^{k_1+k_2} s_1$ and $\langle S, s \rangle \Rightarrow^{k_1+k_2} s_2$. The hypothesis of the rule must be of the form $\langle S, s \rangle \Rightarrow^{k_1} \langle S', s' \rangle$ and $\langle S', s' \rangle \Rightarrow^{k_2} s_1$ or $\langle S', s' \rangle \Rightarrow^{k_2} s_2$. Follows by the inductive hypothesis $s_1 = s_2$.

□

Theorem 4.2. $\langle S, s \rangle \Rightarrow^k s'$ and $\langle S, s \rangle \Rightarrow^k \langle S', s'' \rangle$ can not happen at the same time.

Proof. By induction on k .

- By the corresponding rule $\langle as := a, s \rangle \rightarrow^{Time(a)s} s[ax := Value(a)s]$ and is not possible $\langle as := a, s \rangle \rightarrow^k \langle S, s' \rangle$ for any k, S and s' .
- With $\langle skip, s \rangle$ is similar.
- Consider the conclusion of the first rule for composition: $\langle S_1; S_2, s \rangle \Rightarrow^k \langle S'_1; S_2, s' \rangle$ By hypothesis $\langle S'_1; S_2, s' \rangle \Rightarrow^k \langle S_2, s''' \rangle$ or $\langle S'_1; S_2, s' \rangle \Rightarrow^k \langle S''_1; S_2, s'' \rangle$ but not both, so $\langle S_1; S_2, s \rangle \Rightarrow^k \langle S_2, s''' \rangle$ or $\langle S_1; S_2, s \rangle \Rightarrow^k \langle S''_1; S_2, s'' \rangle$ but not both.
- The second rule for composition is similar.
- By the **if1** rule $\langle if\ b\ then\ S_1\ else\ S_2, s \rangle \Rightarrow^{Time(b)s+1} \langle S_1, s \rangle$. By hypothesis $\langle S_1, s \rangle \Rightarrow^k s'$ or $\langle S_1, s \rangle \Rightarrow^k \langle S'_1, s'' \rangle$ but not both. Then $\langle if\ b\ then\ S_1\ else\ S_2, s \rangle \Rightarrow^{Time(b)s+1+k} s'$ or $\langle if\ b\ then\ S_1\ else\ S_2, s \rangle \Rightarrow^{Time(b)s+1+k} \langle S'_1, s'' \rangle$ but not both.

- The rule **if2** is likely.
- The rule **while** is likely.
- Consider the transitivity rules. Can happen the following cases: $\langle S, s \rangle \Rightarrow^k \langle S', s' \rangle$ and $\langle S', s' \rangle \Rightarrow^{k'} \langle S'', s'' \rangle$ or $\langle S', s' \rangle \Rightarrow^{k'} s''$. Depending on the last transitions we chose rule **trans1** or **trans2**. The other possibility is $\langle S, s \rangle \Rightarrow^k s'$ case in which this is the conclusion.

□

Corollary 4.3. *If $\langle S, s \rangle \Rightarrow^{k_1} \langle S', s' \rangle$ and $\langle S, s \rangle \Rightarrow^{k_2} s''$ then $k_1 \leq k_2$.*

Corollary 4.4. *If $\langle S, s \rangle \Rightarrow^{k_1} \langle S', s' \rangle$ and $\langle S, s \rangle \Rightarrow^{k_2} s''$ then $k_1 \leq k_2$.*

Proof. suppose $k_2 < k_1$. Then exists states s', s'' and a statement S' such that $\langle S, s \rangle \Rightarrow^{k_2} s''$ and $\langle S, s \rangle \Rightarrow^{k_2} \langle S', s' \rangle$ what by the previous theorem can not happen. □

Corollary 4.5 (S is a function). *If $\langle S, s \rangle \Rightarrow^{k_1} s'$ and $\langle S, s \rangle \Rightarrow^{k_2} s''$ then $k_1 = k_2$ and $s' = s''$.*

Proof. Suppose $k_1 \neq k_2$, in particular $k_1 < k_2$. Then happens $\langle S, s \rangle \Rightarrow^{k_1} s'$ and $\langle S, s \rangle \Rightarrow \langle S', s' \rangle$ what is absurd. □

5 Equivalence with the structural operational semantics

5.1 Structural operational semantics

See [3] chapter 2 section 2 for the structural operational semantics of the language.

5.2 Equivalence

Lemma 5.1. *If $\langle S, s \rangle \Rightarrow^* s'$ then there exists k such that $\langle S, s \rangle \Rightarrow^k s'$*

Proof. The rules of the structural operational semantics are included in the measured semantics with the exception that in the last, we have, the measure of time and transitivity rules. In the structural operational semantics the transitive closure is made outside of the rules, but has the same result. If we reach a final state in the structural operational semantics, the same state is reached in the measured semantics. □

Lemma 5.2. *If $\langle S, s \rangle \Rightarrow^k s'$ then $\langle S, s \rangle \Rightarrow^* s'$*

Proof. For each application of a rule in the measured semantics, apply the corresponding one step rule in the structural operational semantics. □

Theorem 5.3 ($\mathcal{S}_{sos} = \mathcal{S}_{stos}$). .

Proof. Lemmas 5.1 and 5.2. □

6 Equivalence with the natural semantics

6.1 Natural semantics

See [3] chapter 6 section 5 for the natural semantics of the language with exact execution times.

6.2 Equivalence

Lemma 6.1. *If $\langle S, s \rangle \rightarrow^k s'$ then $\langle S, s \rangle \Rightarrow^k s'$*

Proof. By induction on the time of the derivation.

- $S = ax := a$. We assume $\langle ax := a, s \rangle \rightarrow^{Time(a)s+1} s[inl(ax) := inl(Value(a)s)]$. Applying *ass*, we get $\langle ax := a, s \rangle \Rightarrow^{Time(a)s+1} s[inl(ax) := inl(Value(a)s)]$.
- $S = skip$ analogous.
- $S = S_1; S_2$. Assume $\langle S_1; S_2, s \rangle \rightarrow^k s''$ because $\langle S_1, s \rangle \rightarrow^{k_1} s'$ and $\langle S_2, s' \rangle \rightarrow^{k_2} s''$ and $k = k_1 + k_2$. Applying the induction hypothesis to the derivations we get $\langle S_1, s \rangle \Rightarrow^{k_1} s'$ and $\langle S_2, s' \rangle \Rightarrow^{k_2} s''$. Applying rule *comp2* to $\langle S_1, s \rangle \Rightarrow^{k_1} s'$ we derive $\langle S_1; S_2, s \rangle \Rightarrow^{k_1} \langle S_2, s' \rangle$, and applying *trans2* to this conclusion and $\langle S_2, s' \rangle \Rightarrow^{k_2} s''$ we get $\langle S_1; S_2, s \rangle \Rightarrow^{k_1+k_2} s''$.
- $S = if\ b\ then\ S_1\ else\ S_2$. Assume that $\langle if\ b\ then\ S_1\ else\ S_2, s \rangle \rightarrow^{Time(b)s+k+1} s'$, because $B[b]s = true$ and $\langle S_1, s \rangle \rightarrow^k s'$. By induction hypothesis applied to $\langle S_1, s \rangle \rightarrow^k s'$ we get $\langle S_1, s \rangle \Rightarrow^k s'$. Applying *if1* to S and *trans2* we get $\langle if\ b\ then\ S_1\ else\ S_2, s \rangle \Rightarrow^{Time(b)s+k+1} s'$. If $B[b]s = false$ is analogous.
- $S = while\ b\ do\ S'$. Assume that $\langle while\ b\ do\ S', s \rangle \rightarrow^k s'$ because $B[b]s = true$ and $\langle S', s \rangle \rightarrow^{k_1} s'$ and $\langle while\ b\ do\ S', s' \rangle \rightarrow^{k_2} s''$. By induction hypothesis $\langle S', s \rangle \Rightarrow^{k_1} s'$ and $\langle while\ b\ do\ S', s' \rangle \Rightarrow^{k_2} s''$. Applying *while*, *if1*, and *comp2* we get:
 $\langle while\ b\ do\ S', s \rangle \Rightarrow^{Time(b)s+2} \langle S'; while\ b\ do\ S', s \rangle \Rightarrow^{k_1} \langle while\ b\ do\ S', s' \rangle$
 applying *trans2* two times we get $\langle S'; while\ b\ do\ S', s \rangle \Rightarrow^{k_1+k_2} s''$ finally applying *trans2* to both derivations, we get $\langle while\ b\ do\ S', s \rangle \Rightarrow^{Time(b)s+k_1+k_2+2} s''$.
- $S = while\ b\ do\ S'$. Assume that $\langle while\ b\ do\ S', s \rangle \rightarrow^{Time(b)s+3} s$ because $B[b]s = false$. Then $\langle while\ b\ do\ S', s \rangle \Rightarrow^{Time(b)s+2} \langle skip, s \rangle$, i.e. $\langle while\ b\ do\ S', s \rangle \Rightarrow^{Time(b)s+3} s$.

□

Lemma 6.2. *If $\langle S_1; S_2, s \rangle \Rightarrow^k s''$ ($k \geq 2$), then there exists a state s' and positive integers k_1 and k_2 such that $\langle S_1, s \rangle \Rightarrow^{k_1} s'$ and $\langle S_2, s' \rangle \Rightarrow^{k_2} s''$ where $k = k_1 + k_2$.*

Proof. See Lemma 2.19 in [3]. For our semantics the proof is analogous. □

Lemma 6.3. *If $\langle S, s \rangle \Rightarrow^k s'$ then $\langle S, s \rangle \rightarrow^k s'$*

Proof. By induction on k .

- $S = ax := a$. Holds because $\langle ax := a, s \rangle \rightarrow^{Time(a)s+1} s[inl(ax) := inl(Value(a)s)]$.
- $S = skip$ is similar.
- $S = S_1; S_2$. Assume that $\langle S_1; S_2, s \rangle \Rightarrow^k s''$. Applying 6.2 we know that there exists s' , k_1 and k_2 such that $\langle S_1, s \rangle \Rightarrow^{k_1} s'$ and $\langle S_2, s' \rangle \Rightarrow^{k_2} s''$ where $k_1 + k_2 = k$. Follows $k_1 < k$ and $k_2 < k$.

By induction hypothesis, there are derivations in the natural semantics for $\langle S_1, s \rangle \rightarrow^{k_1} s'$ and $\langle S_2, s' \rangle \rightarrow^{k_2} s''$. Using the rule in the natural semantics for composition we get $\langle S_1; S_2, s \rangle \rightarrow^k s''$.

- $S = \text{if } b \text{ then } S_1 \text{ else } S_2, \text{Value}(b)s = \text{true}$. Assume $\langle \text{if } b \text{ then } S_1 \text{ else } S_2, s \rangle \Rightarrow^k s'$, we must have transitions $\langle \text{if } b \text{ then } S_1 \text{ else } S_2, s \rangle \Rightarrow^{\text{Time}(b)s+1} \langle S_1, s \rangle$ and $\langle S_1, s \rangle \Rightarrow^{k-\text{Time}(b)s-1} s'$. In the natural semantics we have the transition $\langle S_1, s \rangle \rightarrow^{k-\text{Time}(b)s-1} s'$ and $\langle \text{if } b \text{ then } S_1 \text{ else } S_2, s \rangle \rightarrow^k s'$.

The case $\text{Value}(b)s = \text{false}$ is analogous.

- $S = \text{while } b \text{ do } S'$. Suppose we have $\langle \text{while } b \text{ do } S', s \rangle \Rightarrow^k s', \text{Value}(b)s = \text{true}$.

The derivation in our semantics is

$$\langle \text{while } b \text{ do } S', s \rangle \Rightarrow^1 \langle \text{if } b \text{ then } (S'; \text{while } b \text{ do } S') \text{ else skip}, s \rangle$$

and we must have a derivation for

$$\langle \text{if } b \text{ then } (S'; \text{while } b \text{ do } S') \text{ else skip}, s \rangle \Rightarrow^{k-1} s' \text{ that reduces to } \langle \text{if } b \text{ then } (S'; \text{while } b \text{ do } S') \text{ else skip}, s \rangle \Rightarrow^{\text{Time}(b)s+1} \langle S'; \text{while } b \text{ do } S' \rangle, s \rangle.$$

We have in the natural semantics

$$\langle \text{while } b \text{ do } S', s \rangle \rightarrow^{\text{Time}(b)+t'+t''+2} s'', \text{ if } \langle S', s \rangle \rightarrow^{t'} s' \text{ and } \langle \text{while } b \text{ do } S', s' \rangle \rightarrow^{t''} s'',$$

$$\langle \text{if } b \text{ then } (S'; \text{while } b \text{ do } S') \text{ else skip}, s \rangle \rightarrow^{\text{Time}(b)s+t+1} s' \text{ where } t = t' + t' \text{ and } \langle S'; \text{while } b \text{ do } S' \rangle, s \rangle \rightarrow^t s'. \text{ By Lemma 6.3 } \langle S'; \text{while } b \text{ do } S' \rangle, s \rangle \Rightarrow^t s', \text{ then } k-1 = 1+t+\text{Time}(b)s$$

follows $k-1 = \text{Time}(b)s + t + 1$ then $k = \text{Time}(b)s + t + 2$.

- $S = \text{while } b \text{ do } S'$. Suppose we have $\langle \text{while } b \text{ do } S', s \rangle \Rightarrow^k s, \text{Value}(b)s = \text{false}$.

The derivation in our semantics is

$$\langle \text{while } b \text{ do } S', s \rangle \Rightarrow^1 \langle \text{if } b \text{ then } (S'; \text{while } b \text{ do } S') \text{ else skip}, s \rangle$$

And we must have a derivation for

$$\langle \text{if } b \text{ then } (S'; \text{while } b \text{ do } S') \text{ else skip}, s \rangle \Rightarrow^{k-1} s.$$

We have in the natural semantics

$$\langle \text{if } b \text{ then } (S'; \text{while } b \text{ do } S') \text{ else skip}, s \rangle \rightarrow^{\text{Time}(b)s+2} s, \text{ where } \langle \text{skip}, s \rangle \rightarrow^1 s \text{ and } k-1 = \text{Time}(b)s+2 \text{ then it follows } \langle \text{while } b \text{ do } S', s \rangle \rightarrow^k s.$$

□

Theorem 6.4 ($\mathcal{S}_{tns} = \mathcal{S}_{tsos}$). .

Proof. Lemmas 6.1 and 6.3. □

7 Conclusions and further work

We have presented a semantics that takes account of the number of steps of evaluation of expressions and statements following traditional complexity theory.

We expect in a future work, extend the semantics presented in this paper with functions and procedures. We are interested in finding recurrence expressions to measure the time of iterative and recursive programs. Recursivity will give rise to recurrences and in this way can be possible to find equations for the time of a program.

References

- [1] Matthew Hennesy. *The Semantics of Programming Languages, An Elementary Introduction using Structural Operational Semantics*, John Wiley & Sons, Chichester-New york-Brisbane-Toronto-Singapore, 1990.
- [2] Gilles Kahn. *Natural Semantics*, Programming of Future Generation Computers, Elsevier Science Publishers (North-Holland), 1988.
- [3] H.R.Nielson and F.Nielson. *Semantics with Applications, A Formal Introduction for Computer Science*, John Wiley & Sons, 1992.
- [4] Gordon Plotkin. *A structural Approach to operational semantics*, Technical Report DAIMI FN-19, Aarhus University,1981.